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# Homogeneous Quantization and Multiplicities of Group Representations

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Let $B$ be a compact manifold. A cone over $B$ is a principal $\mathbf{R}^+$-bundle, $X$, with base $B$. Let $(a, x) \to \rho_a(x)$ be the mapping associated with the action of $a \in \mathbf{R}^+$ on $X$. $X$ is called a *symplectic cone* if it possesses a symplectic form, $\omega$, such that $\rho_a^* \omega = a\omega$. A compact Lie group, $G$, is said to act in a homogeneous fashion on $X$ if it acts on $X$ in such a way that both $\omega$ and the principal bundle structure are preserved. It is known that to such an action one can associate in a fairly canonical way a representation of $G$ on a Hilbert space $H$. (See [3].) In this paper we propose a symplectic recipe for the multiplicities with which $H$ decomposes into $G$-irreducibles and show that this recipe is correct "generically."

## INTRODUCTION

Consider a classical mechanical system consisting of a finite number of particles, subject to conservative forces and rigid constraints. The state space of such a system is a symplectic manifold, $X$, and the dynamics of the system is given by a one-parameter group of canonical transformations

$$\exp tH_p, \qquad -\infty < t < \infty,$$

where $p: X \to \mathbf{R}$ is the energy function of the system; i.e., if $x \in X$, $p(x)$ is the energy of the system in the state, $x$.

A finite particle quantum mechanical system is described somewhat differently. The state space now consists of the rays in a Hilbert space, $H$, and the dynamics is given by a strongly continuous one-parameter unitary group

$$\exp itP, \qquad -\infty < t < \infty$$

whose infinitesimal generator, $P$, is the energy operator; i.e., the energy of the system in the state $f \in H$ is given by

$$\langle Pf, f\rangle / \langle f, f\rangle.$$



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A finite particle physical system usually has both a "classical" description which provides one with information about its macroscopic behavior ($\hbar$ small) and a "quantum" description which provides one with information about its microscopic behavior ($\hbar = 1$). Since the classical description is often easier to come by, this means in practice that one tries to construct the pair $(H, P)$, in some intrinsic way, from the pair $(X, p)$. There certain ground rules for doing this. For instance suppose that the sets $p \leqslant \lambda$ are all compact. Then one would like the spectrum of $P$ to be discrete, the number, $N(\lambda)$, of eigenvalues less than $\lambda$ to be finite and

$$N(\lambda) \sim \text{volume } (p < \lambda) \tag{1.1}$$

for $\lambda$ large, since the right hand side is a measure of the number of classical states with energy less than $\lambda$ and the left hand side a measure of the number of quantum states less than $\lambda$.

Suppose now that instead of being given a single classical observable, $p$, we are given a finite collection of classical observables $p_1,..., p_m$ whose Poisson brackets satisfy

$$\{p_i, p_j\} = \sum c_{ij}^k p_k, \tag{1.2}$$

the $c_{ij}^k$'s being constants. Quantum mechanically this corresponds to a finite collection of quantum observables, $P_1,..., P_m$, whose operator brackets satisfy

$$[P_i, P_j] = \sum c_{ij}^k p_k. \tag{1.3}$$

Suppose that $\{X, p_1,..., p_m\}$ and $\{H, P_1,..., P_m\}$ are classical and quantum descriptions of the same physical system. Suppose we replace $N(\lambda)$ by a function which in some appropriate way counts the dimensions of the invariant subspaces of $H$. Is there an analogue of (1.1)? We will propose below such an analogue and then devote the rest of this article to establishing its validity in a spacial case.

Before we start, it will be useful to interpret the data above in more invariant, group-theoretic language. Let $\mathfrak{g}$ be the Lie algebra with structure constants $\{c_{ij}^k\}$ and let $\mathscr{P}$ be the Poisson algebra ($C^\infty$ functions on $X$ equipped with the Poisson bracket operation). Then (1.2) can be interpreted as a morphism of Lie algebras

$$\tau: \mathfrak{g} \to \mathscr{P}. \tag{1.4}$$

Let $G$ be a Lie group with $\mathfrak{g}$ as its Lie algebra. An action of $G$ on $X$ is called a *Hamiltonian* action if for all $\xi \in \mathfrak{g}$

$$\exp t\xi = \exp tH_p,$$

where $p = \tau(\xi)$. Given such an action then for each state, $x$, of the system one can define its "momentum," $\Phi(x)$, as an element of the dual space, $\mathfrak{g}^*$, of $\mathfrak{g}$ by the formula

$$\langle \Phi(x), \xi \rangle = \tau(\xi)(x), \qquad \forall \xi \in \mathfrak{g}.$$

The resulting map $\Phi: X \to \mathfrak{g}^*$ is called the *moment mapping*. In coordinates it is just the mapping above, $(p_1, ..., p_m): X \to \mathbb{R}^m$. The advantage of describing it abstractly, as we have just done, is that it is easy to check that it intertwines the action of $G$ on $X$ with the co-adjoint action of $G$ on $\mathfrak{g}^*$.

Identity (1.3) describes a representation of $\mathfrak{g}$ on $H$. Suppose there corresponds to it a unitary representation

$$\rho: G \to U(H).$$

Given an irreducible representation, $\mu$, of $G$, let $N(\mu)$ be the multiplicity with which $\mu$ occurs in $H$. The analogue of (1.1) would be an exact (or asymptotic) formula for $N(\mu)$ in terms of symplectic invariants of $X$. We will first propose such a formula for the case $G = S^1$. In this case the irreducible representations are indexed by $\sqrt{-1}n$, $n \in Z$. The Lie algebra of $S^1$ and its dual can both be identified with $\sqrt{-1}\mathbb{R}$, so the moment map is a mapping

$$\Phi: X \to \sqrt{-1}\mathbb{R}.$$

Let us assume that $\Phi$ is proper and that the lattice points of $\sqrt{-1}R$ are *regular values* of $\Phi$. Then $d\Phi \neq 0$ on the set

$$Z_n = \Phi^{-1}(\sqrt{-1}n),$$

so this set is a compact manifold on which $S^1$ acts in a locally free fashion. We will simplify our situation somewhat by assuming that $S^1$ acts *freely* on $Z_n$. Then

$$X_n = Z_n/S^1$$

is a compact manifold. We will see shortly that it has a canonical symplectic structure. Our conjecture will express the multiplicity, $N_n$, with which the representation, $\sqrt{-1}n$, occurs in $H$ in terms of topological invariants of $X_n$. To describe these invariants, we need review some standard facts about the topology of symplectic manifolds: Every compact symplectic manifold, $X$, possesses an almost-complex structure, $\mathscr{E}$. Moreover, $\mathscr{E}$ is unique up to topological equivalence. (See Steenrod [8, p. 214].) Let $c_i = c_i(\mathscr{E})$ be the $i$th Chern class of $\mathscr{E}$. [8, p. 210].) From the $c_i$'s one constructs the Todd class of $X$ by the following procedure of Hirzebruch. One writes the expression

$$\prod_{i=1}^{n} x_i/(1 - \exp(-x_i))$$

as a function of $w_1, \ldots, w_n$, where $w_i$ is the $i$th symmetric polynomial in $x_1, \ldots, x_n$. Substituting $c_i$ for $w_i$ in this expression one gets a cohomology class, $\tau \in H^*(X, Q)$. This is by definition the *Todd class* of $X$. Now let $[\omega]$ be the cohomology class of the symplectic form of $X$. The characteristic number

$$\tau e^{[\omega]}(X)$$

is called the *Riemann–Roch* number of $X$.

Coming back to our multiplicity conjecture, we conjecture that

$$N_n = \text{the Riemann–Roch number of } X_n \tag{1.5}$$

for all but finitely many $n$'s. This conjecture has been proved in a number of interesting cases by Boutet de Monvel and one of the authors. (See Section 3 below.)

Next, let $G$ be an arbitrary compact Lie group. By the Borel–Weill–Kostant theorem there is a one-one correspondence between the irreducible representations of $G$ and the integral co-adjoint orbits in $\mathfrak{g}^*$. (For instance, for $G = S^1$ this is the correspondence, which we already alluded to, between irreducible representations and lattice points in $\sqrt{-1}\mathbb{R}$.) Given an integral co-adjoint orbit, $O$, let $X_O$ be the corresponding *reduced space.* (See Section 2 below. For instance if $G = S^1$ and $O = \sqrt{-1}n$, $X_O$ is the space $X_n$ defined above.) Modulo some assumptions about the moment mapping, $X_O$ is a compact symplectic manifold. We conjecture that for "generic" $O$'s

$$N_O = \text{the Riemann–Roch number of } X_O, \tag{1.6}$$

$N_O$ being the multiplicity with which the irreducible representations of $G$ corresponding to $O$ occurs in $H$. We proved this conjecture for compact Kaehler manifolds in [11], and will prove it for another class of examples below (and in doing so generalize the result, for $G = S^1$, referred to above).

It is easy to see that for orbits situated far away from the origin

$$\tau e^{[\omega]}(X_O) \sim e^{[\omega]}(X_O) = \text{volume } (X_O),$$

so (1.6) implies the asymptotic relation

$$N_O \sim \text{volume } (X_O). \tag{1.7}$$

For $G = S^1$ this formula is essentially (1.1).

A few words about the organization of the paper: In Section 2 we define the space, $X_O$, and other reduced spaces which we will need later on. This material is mostly a review of material from an earlier article of ours, [12]. In Section 3 we formulate our main result. In Section 4 we review some symplectic technology needed for the proof (again mostly taken from [12].) In Section 5 we prove a rudimentary form of (1.6) and in Section 6 we prove (1.6) for "ladder representations." Finally in Section 7 we prove our main theorem.

## 2. Reduction

Let $G$ be a connected Lie group, $\mathfrak{g}$ its Lie algebra, $f$ an element of $\mathfrak{g}^*$, $O$ the orbit of $G$ through $f$, $G_f$ the stabilizer group of $f$ and $\mathfrak{g}_f$ its Lie algebra. If $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{g}_f$ then at $t=0$

$$\begin{aligned}(d/dt)\langle(\exp t\xi)^* f, \eta\rangle &= (d/dt)\langle f, (\exp t\xi)\,\eta\rangle \\ &= \langle f, [\xi, \eta]\rangle \\ &= -(d/dt)\langle(\exp t\eta)^* f, \xi\rangle = 0\end{aligned}$$

since $(\exp t\eta)^* f = f$. This shows that $\eta$ is in the conormal space to $O$ at $f$. Since $\dim O = \dim \mathfrak{g} - \dim \mathfrak{g}_f$, it follows that $\mathfrak{g}_f$ *is* the conormal space to $O$ at $f$.

Now let $\Theta$ be a $G$-invariant submanifold of $\mathfrak{g}^*$, let $f \in \Theta$ and let $\mathfrak{h}_f$ be the conormal space to $\Theta$ at $f$. It follows from what we have just shown that $\mathfrak{h}_f \subset \mathfrak{g}_f$ and it is clear that $\mathfrak{h}_f$ is $G_f$ invariant; so $[\mathfrak{g}_f, \mathfrak{h}_f] \subset \mathfrak{h}_f$; i.e., $\mathfrak{h}_f$ is an *ideal* in the Lie algebra $\mathfrak{g}_f$.

Let $X$ be a symplectic manifold on which the group $G$ acts in a Hamiltonian fashion, and let

$$\Phi: X \to \mathfrak{g}^*$$

be the associated moment mapping. By differentiating $\Phi$ at $x \in X$ we get a linear mapping

$$d\Phi_x: T_x \to \mathfrak{g}^*. \tag{2.1}$$

On the other hand there is a mapping

$$\kappa: \mathfrak{g} \to T_x \tag{2.2}$$

defined by

$$\kappa(\xi) = (d/dt)(\exp t\xi)\,x \quad \text{at} \quad t=0$$

and an identification

$$T_x \to T_x^* \tag{2.3}$$

by means of the symplectic form. Composing (2.2) with (2.3) we get a mapping

$$\mathfrak{g} \to T_x^*. \tag{2.4}$$

We leave for the reader to check that (2.2) and (2.4) are transposes of each other. Using this fact we will prove

PROPOSITION 2.1. *Suppose $\Phi$ is transversal to $\Theta$. Let $Z = \Phi^{-1}(\Theta)$. Then $Z$ is a co-isotropic submanifold of $X$. Moreover, if $x \in Z$ and $f = \Phi(x)$ the map* (2.2) *maps $\mathfrak{h}_f$ bijectively onto the tangent space to the leaf of the null-foliation through $x$.*

*Proof.* Let $v \in T_x X$. Then

$$\begin{aligned} v \in T_x Z &\Leftrightarrow \langle d\Phi_x(v), \xi \rangle = 0 \qquad \text{for all } \xi \in \mathfrak{h}_f \\ &\Leftrightarrow \langle v, d\Phi_x^t(\xi) \rangle = 0 \qquad \text{for all } \xi \in \mathfrak{h}_f \\ &\Leftrightarrow \ \omega(v, \kappa(\xi)) \ = 0 \qquad \text{for all } \xi \in \mathfrak{h}_f \end{aligned}$$

which proves that

$$\kappa(\mathfrak{h}_f) = (T_x Z)^\perp. \tag{2.5}$$

By transversality

$$\dim (T_x Z)^\perp = \operatorname{codim} Z = \operatorname{codim} \Theta = \dim \mathfrak{h}_f,$$

so by (2.5) $\kappa$ maps $\mathfrak{h}_f$ bijectively onto $(T_x Z)^\perp$. Finally since $\kappa(\mathfrak{h}_f)$ is in the kernel of $d\Phi_x$ it is tangent to $T_x Z$, so $(T_x Z)^\perp \subset T_x Z$; i.e., $Z$ is co-isotropic.
Q.E.D.

Let $H_f$ be the connected Lie subgroup of $G_f$ associated with the Lie subalgebra, $\mathfrak{h}_f$, of $\mathfrak{g}_f$. We will say that $\Theta$ is *proper* if for all $f \in \Theta$, $H_f$ is a closed subgroup of $G_f$. For instance if $\Theta$ consists of a single orbit, then $H_f = G_f$, so $\Theta$ is proper.[1]

Suppose now both that $\Theta$ is proper and $G$ is compact. Let $x \in Z$, let $f = \Phi(x)$ and let $L_x$ be the leaf of the null-foliation through $x$. By Proposition 2.1

$$L_x = H_f x = H_f / H_f(x), \tag{2.6}$$

$H_f(x)$ being the stabilizer group in $H_f$ of $x$. Also by Proposition 2.1, $H_f(x)$ is

[1] If $G$ is compact and semi-simple, it is easy to see that the only *closed* submanifolds of $\mathfrak{g}^*$ which are $G$-invariant and proper are co-adjoint orbits and the manifolds discussed in Example 4 below.

a *finite* subgroup of $H_f$, so the space of leaves of the null-foliation is a Hausdorff $V$-manifold in the sense of Satake;[2] i.e. there exists a $V$-manifold, $X^{\#}_{\Theta}$, and a smooth fiber mapping

$$\pi: Z \to X^{\#}_{\Theta}$$

such that the fibers f $\pi$ are the leaves of the null-foliation. This implies that there exists also a symplectic form, $\omega_{\Theta}$, on $X^{\#}_{\Theta}$ such that

$$\iota^*\omega = \pi^*\omega_{\Theta}, \tag{2.7}$$

$\iota$ being the inclusion map of $Z$ into $X$. Furthermore there is a natural action of $G$ on $X^{\#}_{\Theta}$. It is easy to see that this action is Hamiltonian, and that its moment mapping

$$\Phi_{\Theta}: X^{\#}_{\Theta} \to \mathfrak{g}^*$$

is related to the original moment mapping by the identity

$$\Phi_{\Theta} \circ \pi = \Phi \circ \iota. \tag{2.8}$$

(See [12, Theorem 5.2].) We will call $X^{\#}{}_{\Theta}$ the *reduced space* associated with $\Theta$. Examples of such reduced spaces are the following:

EXAMPLE 1. Let $\Theta = \{0\}$. Then $Z = \Phi^{-1}(0)$ and

$$X^{\#}_{\Theta} = Z/G = X_G.$$

This is the *Marsden–Weinstein* reduction of $X$ with respect to the zero orbit in $\mathfrak{g}^*$. (See [21].)

EXAMPLE 2. Let $\Theta = O =$ a co-adjoint orbit. The corresponding reduced space, $X^{\#}_O$, is the Kazhdan–Kostant–Sternberg reduction of $X$ with respect to $O$. (See [19].)

EXAMPLE 3. Let $O$ be a co-adjoint orbit. Then $O^- = \{f \in \mathfrak{g}^*, -f \in O\}$ is also a co-adjoint orbit; so by Kostant's theorem it is a symplectic manifold, and $G$ acts on it in a Hamiltonian fashion. Hence $G$ acts on the product manifold, $O^- \times X$, in a Hamiltonian fashion. If we reduce with respect to the zero orbit in $\mathfrak{g}^*$ we get a symplectic $V$-manifold, $X_O$, which is called the *Marsden–Weinstein* reduction of $X$ with respect to $O$. For its relation to the reduced space in Example 2 see below.

[2] For the theory of $V$-manifolds, see [22]. If, for all $x$, the group $H_f(x)$ in (2.6) is the trivial group, $X^{\#}_{\Theta}$ is an ordinary manifold.

EXAMPLE 4. Let $O$ be a co-adjoint orbit and let

$$C(O) = \{rf, r \in \mathbf{R}^+, f \in O\}.$$

We will call $C(O)$ the *cone over O*. The reduction of $X$ with respect to $C(O)$ will play an important role in our study of "ladder" representations in Section 6. If $O = C(O)$, $O$ is called nilpotent. Such orbits cannot occur if $G$ is compact (because then $\mathfrak{g}^*$ possesses a positive-definite $G$-invariant inner product). If $f \in O$ then

$$\mathfrak{h}_f = \{\xi \in \mathfrak{g}_f, \langle f, \xi \rangle = 0\}. \tag{2.9}$$

If $O$ is nilpotent, $\mathfrak{h}_f = \mathfrak{g}_f$; otherwise $\mathfrak{h}_f$ is a co-dimension one ideal in $\mathfrak{g}_f$. We will show that this ideal contains the commutator ideal of $\mathfrak{g}_f$. Indeed if $\xi, \eta \in \mathfrak{g}_f$ then

$$\begin{aligned}\langle f, [\xi, \eta] \rangle &= (d/dt)\langle f, (\exp t\xi)\, \eta \rangle \\ &= (d/dt)\langle (\exp t\xi)^* f, \eta \rangle = 0\end{aligned}$$

at $t = 0$ since $\exp t\xi \in G_f$, so

$$\mathfrak{g}_f \supset \mathfrak{h}_f \supset [\mathfrak{g}_f, \mathfrak{g}_f]. \tag{2.10}$$

By (2.10) the map, $\xi \in \mathfrak{g}_f \to 2\pi i \langle f, \xi \rangle$, is an infinitesimal character of the Lie group $G_f$. If it can be extended to a global character

$$\chi_f : G_f \to S^1 \tag{2.11}$$

then $f$ is called an *integral* point of $\mathfrak{g}$, and the orbit, $O$, an *integral* orbit. Since $\mathfrak{h}_f$ is, by definition, the kernel of the infinitesimal character, $d\chi_f$, $H_f$ is the kernel of $\chi_f$, and so it is a closed subgroup of $G_f$. Thus we have proved

PROPOSITION 2.2. $C(O)$ is a proper submanifold of $\mathfrak{g}^*$, providing $O$ is integral.

*Remark.* For compact groups the converse is true. We leave the proof of this fact as an exercise for the reader.

Now let $O$ be an integral orbit, let $\Theta = C(O)$, let $X^\# = X^\#_\Theta$ and let

$$\Psi : X^\# \to \mathfrak{g}^*$$

be the moment mapping, $\Phi_\Theta$. Notice that $\Psi(X^\#) \subset C(O)$. Let $\langle\ ,\ \rangle$ be a $G$-invariant inner product on $\mathfrak{g}^*$ normalized so that $\|f\| = 1$ for all $f \in O$, and let $p$ be the function $\|\Psi\|$. In Section 6 we will need the following.

PROPOSITION 2.3. *The Hamiltonian flow,* $\exp tH_p$, $-\infty < t < \infty$, *is periodic of period* $2\pi$.

*Proof.* We will first of all show that there *is* a natural candidate for this flow, i.e., natural $S^1$ action on $X^{\#}$. If $f \in O$ then from (2.11) one gets an isomorphism $G_f/H_f \to S^1$. If $f'$ lies on the ray through $f$ then $G_{f'} = G_f$ and $H_{f'} = H_f$, so $G_{f'}/H_{f'} \cong S^1$; i.e., for all $f \in C(O)$ there is a canonical isomorphism

$$\chi_f: G_f/H_f \to S^1. \tag{2.12}$$

Now let $y \in X^{\#}$ and $f = \Psi(y)$. Since $H_f$ is contained in the stabilizer goup of $y$, we can, for each $a \in S^1$, define

$$\rho_a(y) = \kappa_b(y), \tag{2.13}$$

where $b = \chi_f^{-1}(a)$ and

$$\kappa: G \times X^{\#} \to X^{\#} \tag{2.14}$$

is the intrinsic action of $G$ on $X^{\#}$. We leave for the reader to check that (2.13) defines an action of $S^1$ on $X^{\#}$. To prove that this is identical with the Hamiltonian action given by $H_p$ we need the following lemma.

LEMMA 2.4. *Let* $f_0 \in 0$ *and let* $\xi$ *be an element of* $\mathfrak{g}_{f_0}$ *satisfying* $\langle f_0, \xi \rangle = 1$. *Let* $\gamma$: $C(O) \to \mathbf{R}$ *be the function,* $f \to \|f\|$ *and let* $\gamma_1$: $C(O) \to \mathbf{R}$ *be the function,* $f \to \langle f, \xi \rangle$. *Then* $d\gamma = d\gamma_1$ *at* $f_0$.

*Proof.* First we will show that if $\tilde{\gamma}$ and $\tilde{\gamma}_1$ are the restrictions of $\gamma$ and $\gamma_1$ to $O$, $d\tilde{\gamma} = d\tilde{\gamma}_1 = 0$ at $f$. We recall (see [20]) that the moment mapping associated with the Hamiltonian action of $G$ on $O$ is just the inclusion mapping, $O \to \mathfrak{g}^*$, so, by (2.1) and (2.4)

$$\xi^{\#} \lrcorner\, \omega_0 = d\tilde{\gamma}_1,$$

$\xi^{\#}$ being the vector field on $O$ corresponding to $\xi$. Since $\xi \in \mathfrak{g}_{f_0}$, $\xi^{\#} = 0$ at $f_0$, so $d\tilde{\gamma}_1 = 0$ at $f_0$. Since $\gamma$ is $G$-invariant, $\tilde{\gamma}$ is constant, so $d\tilde{\gamma} = 0$ at $f_0$, proving our claim. Now $\gamma(f_0) = \gamma_1(f_0) = 1$ and both $\gamma$ and $\gamma_1$ are homogeneous of degree one, so $\gamma = \gamma_1$ on the ray $\{tf_0, t \in \mathbf{R}^+\}$. Thus $d\gamma = d\gamma_1$ at $f_0$ as claimed. Q.E.D.

Let $\Xi$ be the infinitesimal generator of the circle group action defined by (2.13). If $y_0 \in X^{\#}$ and $f_0 = \Psi(y_0)$ then at $y_0$

$$\Xi = d\kappa(\xi),$$

where $\xi$ is any element of $\mathscr{g}_{f_0}$ satisfying $\langle f_0, \xi\rangle = \|f_0\|$, and $\kappa$ is as in (2.2) and (2.14). Therefore, by (2.2) and (2.4)

$$\Xi \lrcorner\, \omega = d\gamma_1 \circ \Psi$$

at $y_0$, $\omega$ being the symplectic form on $X^{\#}$. But $d\gamma = d\gamma_1$ at $f_0$ and $\gamma \circ \Psi = p$, so $\Xi \lrcorner\, \omega = dp$; i.e., $\Xi = Hp$. Q.E.D.

By Proposition 2.3 we get a Hamiltonian action of $G \times S^1$ on $X^{\#}$, the moment mapping for the $S^1$-action being the map $\sqrt{-1}p\colon X^{\#} \to \sqrt{-1}\mathbf{R}$. We leave for the reader to check the following

PROPOSITION 2.5. *If one reduces $X^{\#}$ with respect to the Hamiltonian action of $S^1$ at the point, $\sqrt{-1}c \in \sqrt{-1}\mathbf{R}$, one gets the reduced space, $X^{\#}_{cO}$ of Example* 2, *where*

$$cO = \{cf, f \in O\}.$$

To conclude this section we will describe how the reduced spaces, $X^{\#}_O$ of Example 2 and $X_O$ of Example 3 are related:

PROPOSITION 2.6. *As a symplectic manifold, $X^{\#}_O$ is the product manifold*

$$X^{\#}_O = X_O \times O \tag{2.15}$$

*and the action of $G$ is the product action, $G$ acting trivially on $X_O$ and by its co-adjoint action on $O$.*

*Proof.* See [19].

## 3. HOMOGENEOUS QUANTIZATION

Let $B$ be a compact manifold, and let $(X, \pi)$ be a principal bundle over $B$ with structure group $\mathbf{R}^+$. We will call $X$ a *cone* with base $B$. Given such a cone and given $a \in \mathbf{R}^+$, let $\rho_a\colon X \to X$ be the diffeomorphism of $X$ associated with $a$. Suppose $X$ is also equipped with a symplectic form, $\omega$. We will call $X$ a *symplectic cone* if for every $a \in \mathbf{R}^+$

$$\rho_a^* \omega = a\omega. \tag{3.1}$$

Two important examples of symplectic cones are the following:

1. Let $M$ be a compact manifold and let $X$ be the punctured cotangent bundle, $T^*M - 0$.

2. Let $B$ be the boundary of a compact strictly pseudoconvex domain. Given a point, $x \in B$, let $\Sigma_x$ be the subspace of $T_x^*B$ defined by

$$\eta \in \Sigma_x \Leftrightarrow \langle v, \eta \rangle = 0$$

for all holomorphic vectors, $v$, tangent to $B$ at $x$. $\Sigma_x$ is one-dimensional, and the "inward-pointing" orientation of $B$ provides it with a natural orientation, so

$$\Sigma_x = \Sigma_x^+ \cup \{0\} \cup (-\Sigma_x^+),$$

$\Sigma_x^+$ being the positively-oriented component of $\Sigma_x - \{0\}$. Let $\pi: X \to B$ be the cone over $B$ whose fiber at $x$ is $\Sigma_x^+$. The definition of $X$ provides us with an imbedding

$$\iota: X \to T^*B - 0. \tag{3.2}$$

If $\omega$ is the standard symplectic form on $T^*B - 0$, $\iota^*\omega$ is a symplectic form on $X$ satisfying (3.1).

Now let $X$ be a symplectic cone and $G$ a Lie group. We will show that an action of $G$ on $X$ which preserves both the symplectic structure and the conic structure is Hamiltonian. Indeed there exists a unique vector field $\Xi$ on $X$ such that

$$\exp t\Xi = \rho_s, \qquad s = e^t. \tag{3.3}$$

Let $\alpha = \Xi \rfloor \omega$. For each $\xi \in \mathfrak{g}$ let $\xi^\#$ be the vector field corresponding to it. It is easy to check that the map

$$\xi \in \mathfrak{g} \to \phi_\xi = \langle \alpha, \xi^\# \rangle \tag{3.4}$$

gives the required imbedding (1.4). Note that $\phi_\xi$ is homogeneous of degree one in $\mathbf{R}^+$, so the moment map

$$\Phi: X \to \mathfrak{g}^* \tag{3.5}$$

defined by (3.4) is *homogeneous*. This is a fact which we will make considerable use of below.

Boutet de Monvel and one of the authors developed a "quantum theory" of symplectic cones in the monograph [3]. We will describe some of the highlights of this theory:

A. Suppose one is given a symplectic cone, $X$, a compact Lie group, $G$, and an action of $G$ on $X$ preserving both the symplectic structure and the conic structure. One can manufacture out of this data a Hilbert space, $H$, and a unitary representation, $\rho$, of $G$ on $H$. The construction of $H$ and $\rho$ is rather complicated in general, but is fairly easy to describe for the two examples of symplectic cones mentioned above.

1. If $X = T^*M - 0$, $H$ is the intrinsic $L^2$ space of half-densities on $M$. Let $\mathscr{C}$ be the group of homogeneous canonical transformations of $X$, $UF(M)$ the group of unitary Fourier integral operators on $M$ and

$$\beta \colon UF(M) \to \mathscr{C}$$

the mapping which to each Fourier integral operator associates its underlying canonical transformation. The action of $G$ on $X$ provides us with a morphism of groups

$$\gamma \colon G \to \mathscr{C}$$

and the representation, $\rho$, of $G$ on $H$ is given by a morphism of groups

$$\rho \colon G \to UF(M)$$

satisfying

$$\beta \circ \rho = \gamma. \tag{3.6}$$

2. Let $\Omega$ be a compact strictly pseudoconvex domain and $B$ its boundary. If $X$ as in Example 2 above, $H$ is the Hardy space,

$$H = \{f \in L^2(B), \exists g \in \mathscr{O}(\Omega), g \upharpoonright B = f\}.$$

Let $A \colon L^2(B) \to L^2(B)$ be a Fourier integral operator and let $\psi \colon T^*B - 0 \to T^*B - 0$ be its underlying canonical transformation. One can show that if $A$ reduces $H$, $\psi$ maps $X$ onto itself. Let $UT(B)$ be the group of all unitary operators on $H$ which are of the form $A \upharpoonright H$, $A$ being a unitary Fourier integral operator which reduces $H$. Let $\mathscr{C}$ be the group of homogeneous canonical transformations of $X$. It follows from what we have just observed that there is a morphism of groups

$$\beta \colon UT(B) \to \mathscr{C}.$$

The action of $G$ on $X$ provides us with a morphism of groups

$$\gamma \colon G \to \mathscr{C},$$

and, as in Example 1, the representation, $\rho$, of $G$ on $H$ is given by a morphism of groups $\rho \colon G \to UT(B)$ satisfying (3.6).

B. The pair $(H, \rho)$ is *not* canonical. For instance, in Example 1, $\rho$ can be any smooth homomorphism of $G$ into $UF(M)$ satisfying (3.6). In general, starting with an action, $\kappa$, of $G$ on $X$ which preserves both the conic and symplectic structures, one can construct from $\kappa$ a large family of pairs, $(H, \rho)$, all of which are equally eligible, from the micro-local point of view, to be the "quantization" of $\kappa$. We will call any such pair an *admissible* representation. If $G$ is semi-simple one can prove that any two admissible representations are equivalent in the following sense.

DEFINITION 3.1. $(H, \rho) \sim (H', \rho') \Leftrightarrow$ there exists a closed, finite co-dimension, $G$-invariant subspace, $H_1$, of $H$, and a similar subspace, $H_1'$, of $H'$ such that $\rho \upharpoonright H_1$ is unitarily equivalent to $\rho' \upharpoonright H_1'$.

In particular, if $G$ is semi-simple and $\mu$ is an irreducible representation of $G$, then, except for a finite number of $\mu$'s, the multiplicity, $N(\mu)$, with which $\mu$ occurs in the admissible representation $(H, \rho)$ is the same for all $(H, \rho)$. Therefore, it makes sense to look for a dimension function, $N_0$, on the unitary dual, $\hat{G}$, of $G$ such that for every admissible representation $(H, \rho)$

$$N(\mu) = N_0(\mu) \tag{3.7}$$

for all but finitely many $\mu \in \hat{G}$.

C. If $G$ is not semi-simple there can be many inequivalent admissible representations. For example, let $G = S^1$ and let $(H, \rho)$ be an admissible representation. One can manufacture a family

$$(H^{(n)}, \rho^{(n)}) \tag{3.8}$$

of representations as follows. For each integer, $k$, let

$$H_k = \{f \in H, \rho(e^{i\theta}) f = e^{ik\theta} f\}.$$

Now define a new representation $\rho^{(n)}$ on $H = H^{(n)}$ by setting

$$\rho^{(n)}(e^{i\theta}) f = e^{i(k+n)\theta} f$$

on $H_k$. It turns out that each of these representations is admissible, and usually these representations *will not* be equivalent in the sense of Definition 3.1. The following, however, was proved in [3].

THEOREM 3.2. *If* $G = S^1$, *then every admissible representation is equivalent to one of the representations* (3.8).

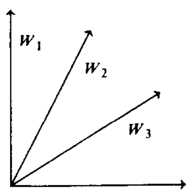
A companion theorem to this was also proved in [3].

THEOREM 3.3. *If* $G = S^1$, *there exists a representation on list* (3.8) *whose multiplicities are given by* (1.5).

If $G$ is semi-simple, our candidate for a dimension function satisfying (3.7) is function (1.6). Unfortunately (1.6) does not even make sense if the reduced space $X_O$ cannot be defined (for instance, if the moment mapping is not transversal to $O$). We will show, however, that (1.6) *is* the correct dimension function for a large number of representations. To state a precise result we must first recall some properties of the moment mapping: Let $X$ be an arbitrary symplectic cone, $G$ a compact semi-simple Lie group, $\kappa: G \times X \to X$ an action of $G$ on $X$ preserving the conic and symplectic structures and $\Phi: X \to \mathfrak{g}^*$ the moment map. As we pointed out above, $\Phi$ is *homogeneous*; it intertwines the $R^+$-action on $X$ and the linear $R^+$-action on $\mathfrak{g}^*$. Since the base of $X$ is compact, it follows that $\Phi$ is *proper* except perhaps at the origin. It also follows that the set, $C$, of critical values of $\Phi$ is a conic subset of $\mathfrak{g}^*$. To describe $C$, we recall that by Weyl's theorem every co-adjoint orbit intersects the positive Weyl chamber, $t_+^*$, in one and exactly one point, so to describe $C$ it is enough to describe its intersection with $t_+^*$. For the following see [10].

THEOREM 3.4. *$C \cap t_+^*$ is a polyhedral cone.*

For instance if rank $G = 2$, $C \cap t_+^*$ looks like the slanted lines in the figure below:




In particular, $\mathrm{Int}(t_+^* - C)$ consists of a finite number of open connected wedges: $W_1, \ldots, W_N$. We will call these wedges the *fundamental wedges* associated with the action of $G$ on $X$. They correspond to the topologically distinct types of reduced spaces. In fact:

THEOREM 3.5. *If the co-adjoint orbits, $O$ and $O'$, intersect $t_+^*$ in the same fundamental wedge, $X_O$ and $X_{O'}$ are diffeomorphic as $V$-manifolds.*

*Proof.* Let $T$ be the Cartan subgroup of $G$. If $f \in \mathrm{Int}(t_+^*)$, $G_f = T$, so $T$ preserves the set

$$Y = \Phi^{-1}(f).$$

If $f \notin C$ then $Y$ is a compact submanifold of $X$, $T$ acts in a locally free fashion on $Y$ and

$$X_O = Y/T \tag{3.9}$$

(see [19]). If $f$ and $f'$ lie in the same connected component of $\text{Int}\, t^*_+ - C$, then, by the "isotopy theorem," (see [1]) $Y$ and $Y'$ are isomorphic as $T$ manifolds, so $X_O \cong X_{O'}$ as $V$-manifolds. Q.E.D.

*Remark.* Some unpublished results of Cushman and Heckman indicate that the topological type of $X_O$ changes dramatically as $f$ goes from one fundamental wedge to another.

If $f$ and $f'$ both belong to the same fundamental wedge, $T$ acts freely on $Y$ if and only if it acts freely on $Y'$. If a fundamental wedge has this property we will call it an *elementary* fundamental wedge. If $f$ belongs to an elementary fundamental wedge, the reduced space, $X_O$ is, by (3.9), a (non-singular) manifold.

By the Borel–Weil–Kostant theorem, there is a one–one correspondence between the points of $\hat{G}$ and the integral co-adjoint orbits in $\mathfrak{g}^*$. Each orbit intersects $t^*_+$ in a lattice point, so $\hat{G}$ can be identified with the lattice points in $t^*_+$. The main theorem of this paper is the following.

THEOREM 3.6. *Let $(H, \rho)$ be an admissible representation of G. Let W be an elementary fundamental wedge and let V be a closed conic subset of $t^*_+ - O$ contained in W. Then for all but a finite number of lattice points in V, the multiplicity, $N(\mu)$, with which $\mu$ occurs in $(H, \rho)$ is given by* (1.6), *O being the orbit through $\mu$.*

To avoid introducing too much micro-local machinery we will only prove this theorem when $X = T^*M - 0$, $H = L^2(M)$ and $\rho: G \to UF(M)$ is a representation of $G$ on $L^2(M)$ by unitary Fourier integral operators. The two main steps in the proof are the following:

*Step* 1. (See Section 5.) We will first prove a general result which says that the multiplicities, $N(\mu)$, for $\mu \in V$ are given by a polynomial function of $\mu$ for $\mu$ sufficiently large.

*Step* 2. (See Section 6.) We will show how to reduce the computation of the multiplicites along a ray $\{k\mu,\ k = 1, 2, ...\}$ to a computation which only involves the group, $S^1$. This step is closely connected with the theory of "ladder representations" which is an interesting subject in its own right.

The same symplectic ideas come into both steps of this proof. We will discuss these ideas in Section 4.

## 4. The Moment Lagrangian

In this section we will review a number of facts about group actions on symplectic manifolds. For brevity we will mostly omit proofs. A more detailed discussion of the material below can be found in [12] or [27]. A notational convention which we will use frequently in this section is the following: If $X$ is a symplectic manifold and $\omega$ its symplectic form we will denote by $X^-$ the symplectic manifold, $X$, with symplectic form, $-\omega$.

Now let $M$ be a compact manifold and $\kappa: G \times M \to M$ an action of $G$ on $M$. Let $X = T^*M - 0$. The graph of $\kappa$,

$$\{(g, m, gm), g \in G, m \in M\}, \tag{4.1}$$

is a smooth submanifold of $G \times M \times M$, so its conormal bundle

$$\Gamma' = \{(g, \gamma, m, \mu, m', \mu'), m' = gm, (\gamma, \mu) = -(d\kappa)^t_{g,m} \mu'\} \tag{4.2$'$}$$

is a Lagrangian submanifold of $T^*G \times X \times X$, and

$$\Gamma = \{(g, \gamma, m, \mu, m', \mu'), m' = gm, (\gamma, \mu) = (d\kappa)^t_{g,m} \mu'\} \tag{4.2}$$

is a canonical relation in $T^*G \times (X \times X^-)$.

Let $L^2(M)$ be the Hilbert space of half-densities on $M$. From the action of $G$ on $M$ one gets a unitary representation, $\rho$, of $G$ on $L^2(M)$. We will prove the following result as motivation for a much more sophisticated result which we will describe (but not prove) in Section 5.

THEOREM 4.1. *Let $K = K(g, m_1, m_2)$ be the Schwartz kernel of the operator, $\rho(g)$, viewed as a distributional function on $G \times M \times M$. Then $K$ is a Lagrangian distribution associated with the Lagrangian manifold* (4.2)$'$.

*Proof.* $K(g, m_1, m_2)$ is just a "delta function" supported on (4.1) and is, therefore, very trivially, such a Lagrangian distribution. Q.E.D.

Weinstein noticed in [26] that (4.2) is the special case of a much more general symplectic object. Namely, let $X$ be a symplectic manifold, $\kappa: G \times X \to X$ a Hamiltonian action and $\Phi: X \to \mathfrak{g}^*$ the associated moment mapping. Make the identification

$$T^*G \cong G \times \mathfrak{g}^*, \tag{4.3}$$

where

$$(g, \gamma) \in T^*G \leftrightarrow (g, (dR_g)^t\gamma) \in G \times \mathfrak{g}^*.$$

With this notation one has

THEOREM 4.2. *The set*

$$\Gamma = \{(g, \gamma, x, gx), \gamma = \Phi(x)\} \tag{4.4}$$

*is a canonical relation in $T^*G \times (X \times X^-)$.*

*Proof.* See [26, p. 21].

We leave for the reader to check that (4.2) is a special case of (4.4). We will call (4.4) the *moment Lagrangian.*

The moment Lagrangian is closely related to another symplectic object which we will now define. Let $\Theta$ be a proper $G$-invariant submanifold of $\mathfrak{g}^*$, and let

$$\Lambda_\Theta = \{(g,f), f \in \Theta, g \in H_f\}. \tag{4.5}$$

By means of (4.3), $\Lambda_\Theta$ can be identified with a subset of $T^*G$.

THEOREM 4.3. *The set, (4.5), is a Lagrangian submanifold of $T^*G$. Moreover, it is invariant with respect to the adjoint action of $G$ on $T^*G$.*

*Proof.* See [13]. We will call a Lagrangian submanifold of $T^*G$ which is invariant with respect to the adjoint action of $G$ *central.* For a detailed study of such manifolds see [13].

Two special cases of (4.5) will be important for us. If $\Theta$ is a co-adjoint orbit, $O$, then

$$\Lambda_O = \{(g,f), f \in O, g \in G_f\}, \tag{4.6}$$

where $G_f$ is the stabilizer group of $f$. The homogeneous variant of this is

$$\Lambda_{\Theta'} \qquad \Theta = C(O), \tag{4.7}$$

where $O$ is an *integral* co-adjoint orbit. It is clear from (4.5) that (4.7) is a *homogeneous* Lagrangian submanifold of $T^*G - 0$.

If $X$ and $Y$ are symplectic manifolds, $\Gamma$ a canonical relation in $X \times Y$ and $\Lambda$ a Lagragian submanifold of $Y$, then the set

$$\Gamma \circ \Lambda = \{x \in X, \exists y \in \Lambda, (x,y) \in \Gamma\} \tag{4.8}$$

is very often a Lagrangian submanifold of $X$. For instance it is an immersed Lagrangian manifold providing the mappings $\tau: \Gamma \to Y$, $\tau(x,y) = y$ and $\iota: \Lambda \to Y$, $\iota(y) = y$ intersect *cleanly.* (See [6, Sect. 5].) Using this fact one can often use the canonical relation, $\Gamma$, to construct new and interesting Lagrangian manifolds in $X$ and $Y$, starting with rather simple ones. We will give a few instances of this using for $\Gamma$ the moment Lagrangian (4.4).

As above let $\kappa: G \times X \to X$ be a Hamiltonian action of $G$ on $X$ and let $\Gamma$ be the moment Lagrangian. The diagonal, $\Delta$, is a Lagrangian submanifold of $X \times X^-$; so we can form (4.7) with $\Lambda = \Delta$. Under appropriate cleanliness hypotheses $\Gamma \circ \Delta$ is a Lagrangian submanifold of $T^*G$. We will call it the *character* Lagrangian associated with the action of $G$ on $X$. If $\Phi: X \to \mathfrak{g}^*$ is the moment mapping, then by (4.4)

$$\Gamma \circ \Delta = \{(g, \Phi(x)), gx = x\}. \tag{4.9}$$

In particular, let $\Theta$ be the image of $\Phi$ in $\mathfrak{g}^*$. It is clear that $\Theta$ is $G$-invariant. It is easy to see that $\Gamma$ and $\Delta$ intersect cleanly if and only if the rank of $\Phi$ is the same at all points of $X$. In this case $\Theta$ is an immersed submanifold of $\mathfrak{g}^*$. If $\Theta$ happens to be a proper imbedded submanifold of $\mathfrak{g}^*$ then by comparing (4.9) with (4.5) we get

THEOREM 4.4. *The character Lagrangian*, (4.9), *is* $\Lambda_\Theta$, *where* $\Theta$ *is the image of* $\Phi$.

For instance this is the case if $\Theta$ is a single co-adjoint orbit.

Now let $X$ be a symplectic manifold and $Z$ a co-isotropic submanifold of $X$. We will declare two points, $z_1$ and $z_2$, of $Z$ to be *equivalent* ($z_1 \sim z_2$) if they lie on the same leaf of the null-foliation. The set

$$\{(z_1, z_2) \in Z \times Z, z_1 \sim z_2\} \tag{4.10}$$

is an immersed Lagrangian submanifold of $X \times X^-$. (See [27, Sect. 3].) Suppose, in addition, that the null-foliation is *fibrating*; i.e., suppose there exists a Hausdorff manifold, $X^\#$, and a fiber mapping $\pi: Z \to X^\#$ such that the fibers of $\pi$ are the leaves of the null-foliation. Then (4.10) is the fiber product

$$Z \times_\pi Z = \{(z_1, z_2) \in Z \times Z, \pi(z_1) = \pi(z_2)\} \tag{4.11}$$

and is, in particular a closed, imbedded Lagrangian relation in $X \times X$.

We will relate this construction to the reducing construction described in Theorem 2.1. Let $\kappa: G \times X \to X$ be a Hamiltonian action of $G$ on $X$, and let $\Phi: X \to \mathfrak{g}^*$ be its moment map. Let $\Theta$ be a proper $G$-invariant submanifold of $\mathfrak{g}^*$. Suppose that $\Theta$ and $\Phi$ intersect transversally and suppose in addition that for each $x \in X$, with $f = \Phi(x) \in \Theta$, $H_f$ acts freely at $x$. Then by Proposition 2.1 the set

$$Z = \{x \in X, \Phi(x) \in \Theta\}$$

is a co-isotropic submanifold of $X$, the null-foliation is fibrating and $X^\#$ is, by definition, the reduced space corresponding to $\Theta$. Let $\Gamma$ be the moment Lagrangian and $\Gamma^t$ its transpose.

THEOREM 4.5. *Under the assumptions above* $\Gamma^t$ *and* $\Lambda_\Theta$ *intersect cleanly in* $T^*G$. *Moreover*,

$$Z \times_\pi Z = \Gamma^t \circ \Lambda_\Theta. \tag{4.12}$$

*Proof.* See [12, Theorem 6.1].

The two examples of proper submanifolds of $\mathfrak{g}^*$ which we will be interested in below are co-adjoint orbits and cones over integral co-adjoint orbits. Co-adjoint orbits are themselves symplectic manifolds. We will show that cones over integral co-adjoint orbits are very close to being symplectic manifolds:

THEOREM 4.6. *Let $O$ be an integral co-adjoint orbit. Then there exists a symplectic cone, $Y$, a homogeneous symplectic action of $G$ on $Y$ and a free homogeneous symplectic action of $S^1$ on $Y$ commuting with the action of $G$ such that*

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr, "\Phi"] & \\ Y/S^1 & \cong & C(O), \end{tikzcd}$$

*$\Phi$ being the moment mapping.*

*Proof.* Let $f \in O$, and let $\chi_f\colon G_f \to S^1$ be homomorphism (2.11). In the set, $G \times S^1$, identify the points $(g, \omega)$ and $(g', \omega')$ if

$$g' = gh \qquad \text{and} \qquad \omega = \chi_f(h)\,\omega' \tag{4.13}$$

for some $h \in G_f$. By making these identifications in $G \times S^1$, one obtains a set, $B$, and a map $\pi\colon B \to O$, where $O = G/G_f$. It is easy to see that $(B, \pi)$ is a principal $S^1$ bundle over $O$. Let $\partial/\partial\theta$ be the infinitesimal generator of the $S^1$-action on $B$. In [20], Kostant proves the following

THEOREM 4.7. *There exists a unique one-form, $\alpha$, on $B$ such that $\langle \alpha, \partial/\partial\theta \rangle = 1$ and $d\alpha = \pi^*\omega_O$, $\omega_O$ being the symplectic form on $O$.*

Let

$$Y = \{(w, c\alpha_w), w \in B, c \in \mathbf{R}^+\}. \tag{4.14}$$

By definition, $Y$ is a submanifold of $T^*B - 0$. It is easy to see that the restriction of the symplectic form on $T^*B$ to $Y$ defines a symplectic structure on $Y$. We leave for the reader to check that $Y$ has the other properties required of it.

## 5. REPRESENTATIONS BY FOURIER INTEGRAL OPERATORS

Let $M$ be a compact manifold and $G$ a compact semi-simple Lie group. Let $X = T^*M - 0$ and let $G \times X \to X$ be an action of $G$ on $X$ preserving both the symplectic and conic structure. As we pointed out in Section 3, this action is automatically Hamiltonian and the moment map associated with it,

$$\Phi: X \to \mathfrak{g}^*,$$

is homogeneous. We will assume that

$$\Phi(x) \neq 0 \qquad \text{for all } x \in X. \tag{5.1}$$

This condition coupled with the homogeneity implies that $\Phi$ is *proper*. Let $\mathscr{C}$ be the group of homogeneous canonical transformations of $X$. The action of $G$ can be regarded as a morphism of groups

$$\gamma: G \to \mathscr{C}.$$

Let $UF(M)$ be the group of unitary Fourier integral operators on $L^2(M)$ and let

$$\beta: UF(M) \to \mathscr{C}$$

be the morphism of groups which, to each Fourier integral operator, associates its underlying canonical transformation. Let $OPS^\natural$ be the space of skew-adjoint first order pseudodifferential operators on $X$. $OPS^\natural$ is a Lie algebra with respect to the operator bracket and can, in a certain sense, be thought of as the Lie algebra of the group $UF(M)$. For $P \in OPS^\natural$ let $\sigma(P)$ be its symbol.

THEOREM 5.1. *There exists a morphism of groups, $\rho: G \to UF(M)$, and a corresponding morphism of Lie algebras, $d\rho: \mathfrak{g} \to OPS^\natural$, such that*

(a) $\beta \circ \rho = \gamma$.

(b) *If* $P = d\rho(\xi)$ *then* $\rho(\exp t\xi) = \exp tP$.

(c) $\sigma(P) = \phi_\xi$, $\phi_\xi$ *being the* $\xi$th *component of the moment mapping.*

This theorem is a special case of Theorem 5.9A of [3]. It was also proved independently by Alan Weinstein (unpublished). A companion theorem to it is the following.

THEOREM 5.2. *Any two representations having the properties above are equivalent in the sense of Definition* 3.1.

In the course of proving Theorem 5.1, the following somewhat stronger assertion is proved.

THEOREM 5.3. *Let $K = K(g, m, m)$ be the Schwartz kernel of the operator, $\rho(g)$, viewed as a distributional function on $G \times M \times M$. Then $K$ is* *a Lagrangian distribution associated with the moment Lagragian* $\Gamma \subset T^*(G) \times X \times X$.

Let $\Delta_{\mathfrak{g}}$ be the Casimir element in the universal enveloping algebra of $\mathfrak{g}$ and let $\Delta_M = d\rho(\Delta_{\mathfrak{g}})$. $\Delta_M$ is a self-adjoint second order pseudodifferential operator which commutes with $\rho(g)$ for all $g \in G$. Hence every irreducible subspace of $L^2(M)$ is an eigenspace of $\Delta_M$. Moreover, if $V_1$ and $V_2$ are two such spaces and $\Delta_M = \lambda_i I$ on $V_i$, $i = 1, 2$, then $\lambda_1 = \lambda_2$ if the representation of $G$ on $V_1$ is isomorphic to the representation on $V_2$. By part (c) of Theorem 5.1 the symbol of $\Delta_M$ is $\langle \Phi, \Phi \rangle$, where $\langle \ , \ \rangle$ is the negative of the killing form on $\mathfrak{g}^*$, so by (5.1), $\Delta_M$ is *elliptic*. Together with the remarks above this proves

THEOREM 5.4. *Every irreducible representation of G occurs in* $L^2(M)$ *with finite multiplicity.*

Let $\Delta: G \times M \to G \times M \times M$ be the diagonal mapping. By Theorem 5.3 and (5.1) the wave front set of $K(g, m_1, m_2)$ consists of points $((g, \gamma), x_1, x_2)$ with $\gamma \neq 0$, so by [17, Sect. 2.5], the pull-back, $\Delta^*K$, is well-defined as a distribution on $G \times M$. For the same reason the integral

$$\int \Delta^*K(g, m)\, dm = \operatorname{trace} \rho(g) \tag{5.2}$$

is well-defined as a distributional function on $G$. We will denote this distribution by $\chi_\rho$. It is, by definition, the *character* of the representation. It is easy to show that it is equal to the usual Harish–Chandra character of $\rho$. (See [25].) Using functorial properties of wave-front sets with respect to pull-backs and push-forwards (see [8]), one obtains from Theorem 5.3

THEOREM 5.5. *The wave-front set of* $\chi_\rho$ *is contained in the character Lagrangian,* $\Lambda_\rho = \Gamma \circ \Delta$.

As we pointed out in Section 4, $\Lambda_\rho$ is a Lagrangian submanifold of $T^*G$, providing the composition of $\Gamma$ with $\Delta$ is *clean*. In the example above this is not true at all points of $\Gamma$ and $\Delta$; however, we will show that it is true on an interesting open subset. Let us denote by $\mathscr{W}$ the set of all points, $p$, in $\mathfrak{g}^*$ such that the orbit through $p$ intersects $t_+^*$ in an elementary fundamental wedge.

LEMMA 5.6. *Let* $p = (g, \gamma, x, x) \in \Gamma$ *with* $(dL_g)^t\gamma \in \mathscr{W}$. *Then g is the identity element of G and the intersection of* $\Gamma$ *with* $\Delta$ *at p is clean.*

*Proof.* If $\gamma \in \mathscr{W}$ the stabilizer group of $G$ at any point $x \in \Phi^{-1}(\gamma)$ is trivial, so this proves the first assertion. The fact that $\Gamma$ intersects $\Delta$ cleanly at $p$ follows easily from the fact that $d\Phi_x$ is surjective. Q.E.D.

Let

$$(T^*G)_{\mathscr{V}} = \{(g, \gamma) \in T^*G, (dL_g)^t \gamma \in \mathscr{V}\}. \tag{5.3}$$

This set is an open bi-invariant subset of the punctured cotangent bundle of $G$. We recall that a distribution on $G$ is *central* if it is invariant under the adjoint action of $G$ on itself. For instance, all characters of representations (such as $\chi_\rho$) are central. We will now show that $\chi_\rho$ is equal to a *Lagrangian* central distribution on $(T^*G)_{\mathscr{V}}$.

THEOREM 5.7. *Let $\mathscr{O}$ be a conic subset of $T^*G - 0$ properly contained in $(T^*G)_{\mathscr{V}}$. Then there exists a central Lagrangian distribution, $\chi_\rho^0$, associated with the cotangent space to the identity element in $T^*G$ such that*

$$WF(\chi_\rho - \chi_\rho^0) \cap \mathscr{O} = \varnothing. \tag{5.4}$$

*Proof.* Combining Lemma 5.6 with standard facts about pull-backs and push-forwards of Lagrangian distributions (see, for instance, [8, Sect. 6.3]), one can easily show that there exists a Lagrangian distribution with property (5.4). By "averaging" with respect to Ad $G$, it is easy to arrange that it be central. Q.E.D.

By the Borel–Weil–Kostant theorem the elements of the unitary dual, $\hat{G}$, are in one–one correspondence with the integral co-adjoint orbits in $\mathfrak{g}^*$, and these in turn can be identified with the lattice points in $t_+^*$. In particular to every lattice point, $\mu \in t_+^*$, corresponds an irreducible representation of $G$. Let $\chi_\mu$ be the character of this representation. It follows from standard facts about characters (see [5]) that

$$\chi_\rho = \sum N_\rho(\mu)\, \chi_\mu, \tag{5.5}$$

the sum taken over the lattice points in $t_+^*$, $N_\rho(\mu)$ being the multiplicity with which the irreducible representation indexed by $\mu$ occurs in $L^2(M)$. To exploit (5.5) we need to review some standard facts about harmonic analysis on symmetric spaces: Consider $G$ as a homogeneous space with respect to the action of $G \times G$ on $G$. The stabilizer group at the identity element is the diagonal subgroup, $G^\Delta$, of $G \times G$ and, as such, is isomorphic to $G$. Its action on $G$ is the adjoint action, so a central function on $G$ is just a function fixed by $G^\Delta$. For the following see [5].

PROPOSITION 5.8. *The representation of $G \times G$ on $L^2(G)$ is a direct sum of irreducible representations. Each such representation occurs just once, and the representations which occur are precisely the representations of the form $\lambda \boxtimes \lambda^*$, where $\lambda$ is an irreducible representation of $G$. The representation* *space, $V_\lambda \otimes V_\lambda^*$, of each of these representations contains a unique $G^\Delta$-fixed vector, namely, $I_{V_\lambda}$. As an element of $L^2(G)$, this $G^\Delta$-fixed vector is just the character, $\chi_\lambda$. Moreover these characters form an orthonormal basis of the space $L^2(G)_{G^\Delta}$.*

In particular every central function can be written as an infinite sum of characters. If we write the delta-function this way we get the Plancherel formula for the group, $G$,

$$\delta = \sum D(\mu)\, \chi_\mu . \tag{5.6}$$

(Here the characters are indexed by the lattice points in $t_+^*$ as in (5.5).) By the Peter–Weyl theorem

$$D(\mu) = \dim V_\mu , \tag{5.7}$$

$(V_\mu, \rho_\mu)$ being the irreducible representation of $G$ induced by $\mu$.

In addition to Proposition 5.8 we will need a theorem of Harish-Chandra which describes the structure of the ring of bi-invariant pseudodifferential operators on $G$. Let us denote this ring by $OPS_G$. If $P \in OPS_G$ then for each irreducible representation, $\rho_\mu$ of $G$, there exists a constant $N(P, \mu)$ such that

$$P = N(P, \mu)\, I \tag{5.8}$$

on the $G \times G$-irreducible subspace of $L^2(G)$ corresponding to $\rho_\mu \boxtimes \rho_\mu^*$, so this shows that the ring $OPS_G$ is *commutative*. It also shows that for each character $\chi_\mu$

$$P\chi_\mu = N(P, \mu)\, \chi_\mu . \tag{5.9}$$

Now let $T$ be a Cartan subgroup of $G$, $t$ its Lie algebra, $t^*$ the dual space of $t$ and $W$ the Weyl group. Let $S(t^*)$ be the ring of all functions, $p$, on $t^*$ for which there exists, for some integer $m$, an asymptotic expansion

$$p \sim \sum_{i=0}^{\infty} p_{m-i}, \tag{5.10}$$

where $p_{m-i}$ is a smooth homogeneous function of degree $m - i$ on $t^* - 0$. Let $S(t^*)^W$ be the subring of $W$-invariant elements of $S(t^*)$. Let $\alpha_i \in t^*$, $i = 1,\ldots, N$ be the positive roots of the Lie algebra, $\mathfrak{g}$, and let $\beta = (1/2) \sum \alpha_i$. Given a $W$-invariant conic open subset, $U$, of $t^* - 0$, let $\mathring{U}$ be the set of all points, $f$, in $\mathfrak{g}^* - 0$ such that the orbit through $f$ intersects $t^*$ in $U$. Let

$$(T^*G)_U = \{(g, \gamma) \in T^*G,\ (dL_g)^t \gamma \in \mathring{U}\}.$$

PROPOSITION 5.9. *There exists a ring isomorphism*

$$\Psi\colon S(t^*)^W \to (OPS)_G$$

*with the following properties:*

(a) *If the leading term of* (5.10) *is of order* $m$ *then* $P = \Psi(p)$ *is of order* $m$ *and its symbol, restricted to the subset* $\{(e, \gamma), \gamma \in t^*\}$ *of* $T^*G - 0$, *is* $p_m$.

(b) *Let* $p \in S(t^*)^W$ *and let* $P = \Psi(p)$. *If the support of* $p$ *is contained in a conic subset,* $U$ *of* $t^*$, *then the micro-local support of* $P$ *is contained in* $(T^*G)_U$.

(c) *Let* $p \in S(t^*)^W$ *and let* $P = \Psi(p)$. *Then*

$$N(P, \mu) = p(\mu + \beta) \tag{5.11}$$

*for all lattice points* $\mu$ *in* $t_+^*$.

*Proof.* See [14, X, Sect. 6; 15].

If $f$ is a central distribution we can write it as a sum of characters

$$f = \sum a_\mu \chi_\mu. \tag{5.12}$$

It is easy to see that if $f \in C^\infty(G)$ then for all integers $N$

$$a_\mu = O(|\mu|^{-N}).$$

From Proposition 5.9 one immediately gets a micro-local version of this result:

PROPOSITION 5.10. *If* $f$ *is a central distribution and* $WF(f) \cap (T^*G)_U = \varnothing$, *then*

$$a_\mu = O(|\mu|^{-N})$$

*for all lattice points,* $\mu \in U$.

Let $\rho$ be the representation described in Theorem 5.1 and let $N_\rho(\mu)$, as in 5.5, be the multiplicity with which the irreducible representation of $G$ indexed by $\mu$ occurs in $\rho$. Let $m = \dim M - \operatorname{rank} G$. We can now prove the main analytical result of this section:

PROPOSITION 5.11. *Let* $U$ *be a connected conic subset of* $t_+^*$ *which is properly contained in an elementary fundamental wedge. Then there exist* *smooth functions* $Q_m, Q_{m-1}, \ldots$ *on* $U$ *such that* $Q_{m-i}$ *is homogeneous of degree* $m-i$, $Q_m > 0$ *on* $U$ *and*

$$N_\rho(\mu) \sim \sum_{i=1}^{\infty} Q_{m-i}(\mu), \tag{5.13}$$

*for* $\mu$ *a lattice point in* $U$ *with* $|\mu| \gg 0$.

*Proof.* Every Lagrangian distribution, $f$, on $T^*G$ which is associated with the Lagrangian submanifold, $T_e^*$, can be written in the form

$$f = P\delta \text{ modulo } C^\infty,$$

where $\delta$ is the delta function and $P$ is a left invariant pseudodifferential operator. If $f$ is of order $k$,[3] one can take $P$ to be of order $k - N/2$, where $N = \dim G$. Moreover, if the symbol of $f$ is non-vanishing on a subset of $T_e^*$ one can arrange for the same to be true of the symbol of $P$. Finally if $f$ is central then by the "method of averaging" one can arrange for $P$ to be in $(OPS)_G$. If $P = \Psi(P)$ and $f = P\delta$ then by (5.6) and (5.11)

$$f = \sum D(\mu)\, p(\mu + \beta).$$

By the Weyl dimension formula, $D(\mu)$ is a polynomial in $\mu$ of degree $N - r$, where $r = \operatorname{rank} G$. Moreover its leading term is non-zero on $\operatorname{Int}(t_+^*)$. Thus $D(\mu)\, p(\mu + \beta)$ admits an asymptotic expansion in $\mu$ whose leading term is of degree $k + N/2 - r$ and is non-zero when $\mu$ is a regular element of $t^*$ and the symbol of $f$ does not vanish at $\mu$.

Now appy this result to the distribution, $\chi_\rho^0$, in Theorem 5.7. It is easy to see, using the functorial properties of Lagrangian distributions with respect to "clean composition" operations described in [6], that $\chi_\rho^0$ is of order $n - N/2$ and that its symbol is non-vanishing on $U$, so the Fourier coefficients of $\chi_\rho^0$ admit an asymptotic expansion of the form (5.13). But by Proposition 5.10, the $\mu$th Fourier coefficient, $a_\mu$, of $\chi_\rho - \chi_\rho^0$ satisfies

$$a_\mu = O(|\mu|^{-N})$$

for $\mu \in U$, so, on $U$, expansion (5.13) for $\chi_\rho^0$ is also valid for $\chi_\rho$. Q.E.D.

We will now show that, by a simple algebraic argument, one can convert the assertion, (5.13), into a considerably stronger assertion: Let $Z^n$ be the standard lattice in $\mathbf{R}^n$, let $U$ be a connected open conic subset of $\mathbf{R}^n$ and let $Q$ be a smooth function on $U$ which has the property that

$$Q(\mu) \sim \sum Q_{m-i}(\mu)$$

[3] Using the order convention of [8, Chap. 6].

for $\mu \in Z^n \cap U$, $Q_{m-i}$ being a smooth homogeneous function of degree $m-i$ on $U-0$. We will prove the following elementary fact.

PROPOSITION 5.12. *Suppose that $Q$ takes on integer values at all but finitely many lattice points, $\mu \in U$. Then at all but finitely many lattice points $Q$ is equal to a polynomial function of $\mu$.*

*Proof.* For $\mu$ large $Q_{m-i}(\mu)$ is small when $m-i<0$; therefore, since $Q(\mu)$ takes on integer values, these terms cannot contribute to the asymptotic expansion for $\mu$ large and can be dropped. Therefore we can assume $Q=\sum_{i=0}^{m} Q_i$. We will next prove

LEMMA 5.13. *$Q(\mu)$ takes on integer values at all integer points of $U$, and so do the functions, $m!Q_i$, for $i=1,\ldots,m$.*

*Proof.* Fix $\mu \in Z^n \cap U$ and let $P(t)=Q(t\mu)$, i.e.,

$$P(t)=\sum_{i=0}^{m} t^i Q_i(\mu). \tag{5.14}$$

By assumption $P(t)$ takes on integer values at all integer points, $t=k$, with $k$ greater than some $k_0$. It is well-known that every $m$th order polynomial function of one variable with this property has to be of the form

$$\sum_{i=0}^{m} c_i \binom{t}{i},$$

where the $c_i$'s are integers and

$$\binom{t}{i}=(t(t-1)\cdots(t-i+1))/i!.$$

(See [28, p. 233].) Apply this result to (5.14). Q.E.D.

Finally we will prove

LEMMA 5.14. *Let $Q$ be a smooth homogeneous function of degree $m$ on $U$ which takes on integer values at all the lattice points $U\cap Z^n$. Then $Q$ is a polynomial.*

*Proof.* Let $\alpha \in U \cap Z^n$ and $\beta \in Z^n$. Then for all integers $k$ greater than some $k_0$, $k\alpha+\beta \in U$ and

$$Q(k\alpha+\beta)=k^m Q(\alpha+\beta/k) \sim \sum k^{m-i} Q^i(\alpha,\beta), \tag{5.15}$$

where

$$Q^i(\alpha, \beta) = \sum_{|\gamma| = i} Q^{(\gamma)}(\alpha)(\beta^\gamma/\gamma!), \tag{5.16}$$

the sum being over all multi-indices, $\gamma$, of length $i$. Since the left hand side of (5.15) only takes on integer values all the terms on the right with $i > m$ must be zero, so, in particular, in view of (5.16), $Q^{(\gamma)}(\alpha) = 0$ at all lattice points, $\alpha \in U$. Since $Q^{(\gamma)}$ is homogeneous of degree $-1$ when $|\gamma| = m + 1$, this implies $Q^{(\gamma)} \equiv 0$ for $|\gamma| = m + 1$, proving the lemma. In view of what we have already shown, this proves Proposition 5.12 as well.

Applying this result to the asymptotic expansion 5.13 we get the main theorem of this section:

THEOREM 5.15. *Let $U \subset t^*$ be an open connected conic set properly contained in an elementary fundamental wedge. Then there exists a polynomial function, $Q$, on $t^*$ of degree equal to* $\dim M - \operatorname{rank} G$, *such that the* $m$th *homogeneous part of $Q$ is nowhere-vanishing on $U$ and*

$$N_\rho(\mu) = Q(\mu) \tag{5.17}$$

*for all but finitely many lattice points, $\mu \in U$.*

## 6. Ladder Representations

Let $G$ be a compact semi-simple Lie group, $T$ a Cartan subgroup of $G$, $t$ its Lie algebra and $t_+^*$ the positive Weyl chamber in $t^*$. We recall that the irreducible unitary representations of $G$ are in one–one correspondence with the lattice points in $t_+^*$. We will call a sequence of lattice points $L = \{k\alpha, k = 0, 1, \ldots\}$ a *ladder*, and we will call a unitary representation of $G$ a *ladder representation* if all its irreducible subpresentations are indexed by the lattice points on a fixed ladder. We will prove in this section that a Fourier integral operator representation of the type described in Theorem 5.1 can, to a large extent, be decomposed into ladder representations of the same type.

We begin by recalling some standard facts about unitary representations of compact topological groups. Let $G$ be such a group and let $\rho: G \to U(H)$ be a unitary representation of $G$ on a separable Hilbert space $H$. Given $\alpha \in \hat{G}$, an element, $v$, of $H$ is said to "transform according to $\alpha$" if it is contained in an irreducible subspace of $H$, and the representation of $G$ on this subspace is equivalent to the representation indexed by $\alpha$. Let $H_\alpha$ be the closure of the space spanned by all such $v$'s. Then

$$H = \sum_{\alpha \in \hat{G}} H_\alpha \qquad \text{(Hilbert space direct sum).} \tag{6.1}$$

For the following see Dixmier [5, p. 291, Theorem 15.3.2].

PROPOSITION 6.1. *Let $\Pi_\alpha$ be the orthogonal projection of $H$ on $H_\alpha$. Let* $\dim \alpha$ *be the dimension of the representation space on which $\alpha$ is represented, let $\chi_\alpha$ be the character of $\alpha$ and let $dg$ be normalized Haar measure on $G$. Then*

$$\Pi_\alpha = (\dim \alpha) \int \rho(g)\, \chi_\alpha(g^{-1})\, dg. \tag{6.2}$$

COROLLARY 6.2. *Let $G$ be a compact semi-simple Lie group, let $L = \{k\alpha, k = 1, 2,\ldots\}$ be a ladder and let $\Pi_L = \sum \Pi_{k\alpha}$. Then*

$$\Pi_L = \int \rho(g)\, \chi_L(g^{-1})\, dg, \tag{6.3}$$

*where*

$$\chi_L = \sum_{k=1}^{\infty} \dim(k\alpha)\, \chi_{k\alpha}. \tag{6.4}$$

Let $O$ be the co-adjoint orbit through $\alpha$, let $C(O)$ be the cone through $O$ and let $\Lambda_L$ be the *character Lagrangian* corresponding to $C(O)$. (See Section 4.) By Theorem 4.3, $\Lambda_L$ is a homogeneous Lagrangian submanifold of $T^*G - 0$. To express the right hand side of (6.3) more concretely, we will need

THEOREM 6.3. *$\chi_L$ is a Lagrangian distribution associated with the Lagrangian submanifold, $\Lambda_L$, of $T^*G - 0$.*

The proof we will give of this theorem is somewhat circuitous. We will begin by constructing a ladder representation in which each of the representations, $k\alpha \in L$, occurs exactly with multiplicity one: By (2.11) there corresponds to $\alpha \in O$ a homomorphism of the stablizer group of $\alpha$ onto $S^1$. Since $O$ is a transitive $G$-space this homomorphism induces on $G$ a homogeneous Hermitian line bundle, $E \to G$. (See [16].) By the Borel–Weil–Kostant theorem there is a $G$-invariant holomorphic structure on $O$ with respect to which $E$ is a holomorphic line bundle; and the carrier space for the irreducible representation of $G$ associated with $O$ is the space of holomorphic sections of $E$. The dual bundle, $E^*$, is also a holomorphic Hermitian line bundle. Let

$$D(E^*) = \{(x, \xi), x \in O, \xi \in E_x^*, |\xi| \leqslant 1\}$$

and

$$B = \{(x, \xi), x \in 0, \xi \in E_x^*, |\xi| = 1\}. \tag{6.5}$$

$D(E^*)$ is a compact complex domain whose boundary, $B$, is strictly pseudoconvex. (See [3, Sect. 13].) Let $H^2(B)$ be the Hardy space of $B$. (See Section 3, Example 2.) Since all the data above are $G$-invariant, there is a natural action of $G$ on $B$. By (6.5), $B$ is a circle bundle over $O$, so there is also a natural action of $S^1$ on $B$ which commutes with the action of $G$. Moreover the complex structure on $B$ is both $G$ and $S^1$ invariant, so there is a representation, $\rho_L$, of $G$ on $H^2(B)$ and, $\tau_L$, of $S^1$ on $H^2(B)$, and these two representations commute. For the following see [3, p. 108, Lemma 13.14].

Proposition 6.4. *$\rho_L$ is a ladder representation and each of the irreducible representations, $k\alpha \in L$, occurs in $\rho_L$ with multiplicity one. Moreover, on the subspace of $H^2(B)$ corresponding to $k\alpha$*

$$\tau_L(e^{i\theta}) = e^{ik\theta} I. \tag{6.6}$$

We pointed out in Section 3 that there corresponds to $B$ a symplectic cone, $Y \subset T^*B - 0$, whose base is $B$. We leave for the reader to show that $Y$ is identical with the symplectic cone of Theorem 4.6. For the following see [4].

Proposition 6.5. *Let $S$ be orthogonal projection of $L^2(B)$ onto $H^2(B)$. Then $S$ is an elliptic Fourier integral operator of complex type associated with the identity canonical relation in $Y \times Y$.*

Let $\partial/\partial\theta$ be the infinitesimal generator of the action of $S^1$ on $B$ and let $D$ be the differential operator, $(1/\sqrt{-1})(\partial/\partial\theta)$. By (6.6), $D$ is equal to $k$ times the identity on the irreducible subspace of $H^2(B)$ corresponding to $k\alpha$. Let $P(k)$ be the dimension of this subspace. By the Weyl dimension formula, $P(k)$ is a polynomial function of $k$, so the operator on $H^2(B)$ which is equal to $P(k)$ times the identity on this subspace is a *differential* operator, viz., $P(D)$. By Proposition 6.5, $P(D)$ is a Fourier integral operator on $H^2(B)$ associated with the identity canonical transformation on $Y \times Y$. From this one easily deduces

Proposition 6.6. *Let $K = K(g, b_1, b_2)$ be the Schwartz kernel of the operator, $\rho_L(g)\, P(D)\, S$, viewed as a distributional function on $G \times B \times B$. Then $K$ is a Lagrangian distribution of complex type whose underlying Lagrangian submanifold in $T^*G \times Y \times Y$ is the moment Lagrangian associated with the action of $G$ on $Y$.*

Now notice that the distribution $\chi_L$ defined by (6.4) is just the distribution

$$\chi_L = \operatorname{trace} \rho_L(g)\, P(D)\, S. \tag{6.7}$$

Using functorial properties of Lagrangian distributions under clean composition hypotheses (see [3, Sect. 7]), one can show that the right hand side of (6.7) is a Lagrangian distribution on $G$ associated with the Lagrangian submanifold, $\Gamma \circ \Delta$, where $\Gamma$ is the moment Lagrangian in Proposition 6.6 and $\Delta$ is the diagonal in $Y \times Y$. But by Theorem 4.4, $\Gamma \circ \Delta = \Lambda_L$. This concludes the proof of Theorem 6.3.

Now let $M$ be a compact manifold, let $X = T^*M - 0$, let $\kappa: G \times X \to X$ be a homogeneous symplectic action of $G$ on $X$, let $\Phi: X \to \mathfrak{g}^*$ be its moment mapping and let $H = L^2(M)$. By Theorem 5.1 there corresponds to $\kappa$ an admissible representation, $\rho$, of $G$ on $H$. Let $\alpha \in t_+^*$ be a lattice point lying in an elementary fundamental wedge and let $L = \{k\alpha, k = 1, 2, \ldots\}$. We will show that the projection operator (6.3) is a Fourier integral projection operator of the type studied in Section 5 of [9]. As above let $O$ be the co-adjoint orbit through $\alpha$, and let $C(O) = \Theta =$ the cone through $O$. Let $Z = \Phi^{-1}(\Theta)$. By Theorem 2.1, $Z$ is a co-isotropic submanifold of $X$. Moreover, if $z \in Z$ and $f = \Phi(z)$ the leaf of the co-isotropic foliation through $z$ is identical with the orbit of the group, $H_f$, through $z$. Since $\alpha$ lies in an elementary wedge and $f$ is conjugate to $\alpha$ in $\mathfrak{g}^*$ the action of $H_f$ at $z$ is *free*, so the reduced space

$$X^\# = X^\#_\Theta$$

is a (non-singular) symplectic manifold. Since $X$ is a symplectic cone, and the action of $G$ on $X$ is homogeneous, the moment map, $\Phi$, is homogeneous by (3.4), so the reduced space $X^\#$ is itself a symplectic cone. Let $\Phi^\#$: $X^\# \to \mathfrak{g}^\#$ be the moment map associated with the action of $G$ on $X^\#$. By (2.8)

$$\Phi \circ \iota = \Phi^\# \circ \pi, \tag{6.8}$$

where $\iota$ is the inclusion map of $Z$ into $X$ and $\pi: Z \to X^\#$ the co-isotropic fibration. This identity implies that the image of $\Phi^\#$ is equal to $C(O)$.

By Theorem 4.5, the fiber product

$$Z \times_\pi Z = \{(z_1, z_2), \pi(z_1) = \pi(z_2)\}$$

is a closed Lagrangian relation in $Z \times Z$ and is related to the character Lagragian, $\Lambda_L$, by the identity

$$Z \times_\pi Z = \Gamma^t \circ \Lambda_L, \tag{6.9}$$

where $\Gamma \subset T^*G \times X \times X$ is the moment Lagrangian. By (6.3)

$$\pi_L = \int \rho(g)\, \chi_L(g^{-1})\, dg.$$

The Schwartz kernel of the operator on the right can be viewed functorially as a "pull-back" composed with a "push-forward," namely, as

$$\int \Delta_G^*(\bar{\chi}_L \boxtimes K)\, dg,$$

where $K$ is as in Theorem 5.3, $\bar{\chi}_L \boxtimes K$ is the exterior tensor product of the distributions, $\bar{\chi}_L$ and $K$, regarded as a distribution on $G \times G \times X \times X$, and $\Delta_G$ is the diagonal map, $G \times X \times X \to G \times G \times X \times X$. By Theorem 5.3, $K$ is a Lagrangian distribution associated with $\Gamma$ and, by Theorem 6.3, $\chi_L$ is a Lagrangian distribution associated with $\Lambda_L$. Moreover, it is easy to check that the composition of $\Gamma^t$ and $\Lambda_L$ in formula (6.9) is clean, so by the theorem cited above on functorial properties of Lagrangian distributions with respect to clean composition operations, we have proved

THEOREM 6.7. *$\pi_L$ is a Fourier integral operator associated with the canonical relation* (6.9).

From the "clean composition" theorem of [6, Sect. 4] one can easily compute the symbol of $\pi_L$: For each $x \in X^\#$ let $F_x$ be the fiber of $\pi: Z \to X^\#$ lying above $x$, and let $L^2(F_x)$ be the space of $L^2$ half-densities on $F_x$. In Section 5 of [9] we show that the symbol of every Fourier integral operator associated with the canonical relation (6.9) is an object of the following sort: It is a function, $\sigma$, on $X^\#$ which to every point $x \in X^\#$ assigns a smoothing operator $\sigma_x: L^2(F_x) \to L^2(F_x)$. In our case $F_x$ can be identified with $H_f$, where $f = \Phi^\#(x)$. Let $\sigma_f$ be orthogonal projection of $L^2(H_f)$ onto the one-dimensional subspace of *invariant* half-densities.

THEOREM 6.8. *Let $\sigma(\Pi_L)$ be the symbol of the projection operator, $\Pi_L$. Then at $x \in X^\#$*

$$\sigma(\Pi_L) = \sigma_f, \tag{6.10}$$

*where $f = \Phi^\#(x)$.*

Let $\mathscr{R}$ be the ring of all Fourier integral operators associated with the canonical relation (6.9). In the terminology of [9, Sect. 5], we have proved

COROLLARY 6.9. *$\Pi_L$ is an elliptic idempotent in $\mathscr{R}$ of symbolic rank one.*

Now let $H_k$ be the subspace of $H$ spanned by those vectors which transform according to $k\alpha$, and let $H_L$ be the Hilbert space direct sum

$$H_L = \sum_{k=0}^{\infty} H_k.$$

The restriction of the representation, $\rho$, to $H_L$ is, by definition, a ladder representation. For each $g \in G$ the operator representing $g$ on $H_L$ is $\rho(g)\,\Pi_L$, so by Theorem 5.1 and Theorem 6.7 we have proved

COROLLARY 6.10. *The ladder representation, $\rho \upharpoonright H_L$, is a representation of $G$ by Fourier integral operators.*

Let $\tau_L$ be the representation of $S^1$ on $H_L$ defined by

$$\tau_L(e^{i\theta}) = e^{ik\theta} I \qquad \text{on} \quad H_k. \tag{6.11}$$

(Compare with (6.6). It is clear that this representation commutes with $\rho_L$.)

THEOREM 6.11. *The representation, $\tau_L$, is a representation of $S^1$ by Fourier integral operators.*

*Proof.* Let $\partial/\partial\theta$ be the infinitesimal generator of the circle group and let $D = (1/\sqrt{-1})\, d\tau_L(\partial/\partial\theta)$; i.e., $D = kI$ on $H_k$. Let $\Delta_{\mathfrak{g}}$ be the Cassimir element in the universal enveloping algebra of $\mathfrak{g}$, and let $\Delta = d\rho(\Delta_{\mathfrak{g}})$. On $H$, $\Delta$ is an elliptic pseudodifferential operator of order 2 with symbol

$$\sigma(\Delta) = \langle \Phi, \Phi \rangle. \tag{6.12}$$

(See Theorem 5.4 and the discussion preceding it.) Moreover, on $H_k$

$$\Delta = \langle k\alpha, k\alpha + \beta \rangle\, I, \tag{6.13}$$

where $\beta$ is as in Proposition 5.9. (See Jacobson [18, p. 247].) Therefore on $H_L$

$$\Delta + (\langle \beta, \beta \rangle^2 / \langle \alpha, \alpha \rangle) = \langle \alpha, \alpha \rangle (D - (\langle \beta, \beta \rangle / \langle \alpha, \alpha \rangle))^2. \tag{6.14}$$

Combining (6.14) with Seeley's Theorem, [23], on fractional powers of elliptic operators, we obtain

PROPOSITION 6.12. *$D$ is the restriction to $H_L$ of a first order elliptic pseudodifferential operator, $D_0$, on $H$,*

Since the representation, $\tau_L$, is defined by

$$\tau_L(e^{i\theta}) = (\exp i\theta D_0)\, \Pi_L, \tag{6.15}$$

this proves Theorem 6.11. In [9, Sect. 5] we call the ring of operators

$$P\Pi_L, \tag{6.16}$$

where $P$ is a pseudodifferential operator reducing $H_L$, *the ring of pseudodif-*

*ferential operators on $H_L$*. We show that if $P$ reduces $H_L$, then there exists a function $p$ on $X^{\#}$ such that

$$\sigma(P) \circ \iota = p \circ \pi, \tag{6.17}$$

$\iota$ and $\pi$ being as in (6.8). We call this function the symbol of operator (6.16). By (6.12) the symbol of $D_0$ is $\|\Phi\|$, so, by comparing (6.8) and (6.17) we get

THEOREM 6.13. *$D$ is a pseudodifferential operator on $H_L$ and its symbol is $\|\Phi^{\#}\|$.*

Proposition 2.3 says that the bicharacteristic flow associated with the symbol of $D$ is period of period $2\pi$. This, of course, *has* to be the case since the one parameter group of operators, (6.15), is period of period $2\pi$.

## 7. THE MULTIPLICITY FORMULA

We will now prove Theorem 5.15. Let $V$ be a conic subset of $\iota_+^* - 0$ which is properly contained in an elementary fundamental wedge. For each lattice point, $\alpha \in V$, let $N(\alpha)$ be the multiplicity with which the irreducible representation of $G$ indexed by $\alpha$ occurs in $\rho$. Let $O_\alpha$ be the co-adjoint orbit through $\alpha$, let $X_\alpha$ be the Marsden–Weinstein reduction of $X$ with respect to $O_\alpha$ and let $R(\alpha)$ be the Riemann–Roch number of $X_\alpha$. We want to show that

$$R(\alpha) = N(\alpha) \tag{7.1}$$

for all but finitely many $\alpha \in V$. In Section 5 we have proved that for all but finitely many lattice points, $\alpha \in V$, $N(\alpha)$ is a polynomial function in $\alpha$ of degree $m$, where $m = \dim X - \operatorname{rank} G$, and that its homogeneous term of degree $m$ is nowhere zero on $V$. J. J. Duistermaat and Gerrit Heckman have recently proved that a similar assertion is true for $R(\alpha)$:

PROPOSITION 7.1. *$R(\alpha)$ is a polynomial function of degree $m$ on $V$ and its leading homogeneous term is nowhere vanishing.*

*Proof.* See [7].[4]

We will now show that these polynomials are equal along rays except for a rescaling which may depend on the ray. For each lattice point, $\alpha \in V$, let $D(\alpha)$ be the dimension of the irreducible representation of $G$ indexed by $\alpha$. By the Weyl dimension formula $D(\alpha)$ is a polynomial function of $\alpha$ of degree equal to $\dim G - \operatorname{rank} G$. Moreover since $V$ is properly contained in the

[4] Their proof is based on the following beautiful observation: The cohomology class of the symplectic form on $X_\alpha$ varies linearly as $\alpha$ varies linearly in $V$. (All the $X_\alpha$'s, $\alpha \in V$, are *topologically* identical by Theorem 3.5.)

interior of $t_+^*$ the leading homogeneous term of $D$ is bounded away from zero on $V$. Let $\alpha \in V$ be a fixed lattice point and let $L$ be the ladder $\{k\alpha, k = 1, 2, \dots\}$.

LEMMA 7.2. *There exists an integer, $k_0$, depending on $L$ such that $RD(k\alpha) = ND((k - k_0)\,\alpha)$ for all but finitely many $k\alpha \in L$.*

*Proof.* Let $\Pi_L$, $H_L$, $\rho_L$ and $\tau_L$ be as in Section 7. We recall that $\rho_L$ is a representation of $G$ on $H_L$ and $\tau_L$ a representation of $S^1$ on $H_L$. Let $O$ be the co-adjoint orbit containing $\alpha$, $C(O)$ the cone through $O$ and $X^{\#}$ the reduction of $X$ with respect to $C(O)$. We proved in Section 7 that $X^{\#}$ is a symplectic cone and that the groups, $G$ and $S^1$, act on $X^{\#}$ in a homogeneous symplectic fashion. Moreover we showed that if

$$\Phi^{\#}: X^{\#} \to \mathfrak{g}^* \tag{7.2}$$

is the moment mapping associated with the action of $G$ and

$$\Psi: X^{\#} \to \sqrt{-1}\mathbf{R}, \tag{7.3}$$

the moment mapping associated with the action of $S^1$ then

$$\Psi = \sqrt{-1}\,\|\Phi^{\#}\|, \tag{7.4}$$

where $\|\ \|$ is the killing metric on $\mathfrak{g}$. Theorems 6.7 through 6.13 can be summarized by saying that with respect to (7.2) and (7.3), $\rho$ and $\tau$ are *admissible* representations of $G$ and $S^1$, respectively. However, for admissible representations of $S^1$ the multiplicity formula is known to be true modulo some ambiguity about scaling. (See Theorems 3.2 and 3.3.) In particular there exists an integer $k_0$ such that for all but finitely many positive integers, $k$, the multiplicity with which the irreducible representation

$$e^{i\theta} \to e^{i(k-k_0)\theta} \tag{7.5}$$

of $S^1$ occurs in $H_L$ is given by the Riemann–Roch number of the reduced space associated with the point-orbit, $\sqrt{-1}k \in \sqrt{-1}\mathbf{R}$. Since the moment map in question is (7.4) this reduced space is the Kazhdan–Kostant–Sternberg reduction of $X$ with respect to the co-adjoint orbit through $k\alpha$. (See Proposition 2.5.) Denote this orbit by $O_k$, denote by $X_k^{\#}$ the K–K–S reduction of $X$ by $O_k$ and denote by $X_k$ the M–W reduction of $X$ by $O_k$. By Proposition 2.6,

$$X_k^{\#} = X_k \times O_k, \tag{7.6}$$

so the Riemann–Roch number of $X_k^{\#}$ is the product of the Riemann–Roch number of $X_k$ and the Riemann–Roch number of $O_k$. By the Borel–Weil theorem the Riemann–Roch number of $O_k$ is the dimension of the irreducible representation of $G$ indexed by $k\alpha$, so the Riemann–Roch number of (7.6) is $RD(k\alpha)$. On the other hand the multiplicity with which representation (7.5) occurs in $H_L$ is by (6.18) equal to the multiplicity with which the representation of $G$ indexed by $(k-k_0)\,\alpha$ occurs in $H_L$ times $\dim((k-k_0)\,\alpha)$; i.e., $ND((k-k_0)\,\alpha)$. Thus these two quantities are equal for all but finitely many $k$. Q.E.D.

We will now prove that $RD = ND$ (and hence $R = N$) at all but a finite number of lattice points in $V$. Let $P$ and $Q$ be polynomials on $V$ and $C$ a constant greater than zero such that $P(\alpha) = RD(\alpha)$ and $Q(\alpha) = ND(\alpha)$ for all lattice points, $|\alpha| > C$. We can also choose $C$ so large that for all points $v \in V$ with $|v| > C$ and all $t \in \mathbf{R}$,

$$P(tv) = Q(0) \Rightarrow |t| < 1. \tag{7.7}$$

(This is because the leading homogeneous term of $P$ is nowhere vanishing on $V$.) If $\alpha \in V$ is a lattice point, then, by Lemma 7.2, there exists an integer, $k_0$, depending on $\alpha$ such that

$$P(k\alpha) = Q((k-k_0)\,\alpha)$$

for all but finitely many positive integers, $k$. Since $P(t\alpha)$ and $Q((t-k_0)\,\alpha)$ are polynomial functions of $t$, this implies that

$$P(t\alpha) = Q((t-k_0)\,\alpha)$$

identically in $t$; in particular $P(k\alpha) = Q(0)$. Hence if $|\alpha| > C$, $k_0 = 0$ by (7.7) and $P(\alpha) = Q(\alpha)$. Thus $P$ and $Q$ are identical.

## 8. Concluding Remarks

1. Let $O$ be a co-adjoint orbit in $\mathfrak{g}^*$. If $O$ intersects the positive Weyl chamber in a non-elementary fundamental wedge, the reduced space, $X_O$, is no longer a non-singular symplectic manifold; however, it is a symplectic $V$-manifold in the sense of Satake. Results of Atiyah and Singer [2] and Kawasaki [29] suggest a natural candidate for the Riemann–Roch number of such a manifold. We conjecture that our multiplicity formula is still true at the lattice point on the non-elementary wedges with this Riemann–Roch number on the right hand side of (1.6).

2. Using Theorem 5.15 and an orbit version of the Frobenius reciprocity theorem (see [19, p. 498]), one can give a new proof of many of the multiplicity results of [11].

3. Let $G$ be a non-compact semi-simple Lie group, $K$ its maximal compact subgroup and $\rho$ an irreducible unitary representation of $G$. $\rho$ is called a ladder representation if $\rho \upharpoonright K$ is a ladder representation in the sense of Section 6. There are a number of interesting examples of such representations. (For instance the metaplectic representation is of this type.) Also one can very often realize ladder representations as subrepresentations of principal series representations. In [9] we constructed such a realization of the metaplectic representation using the micro-local machinery of Section 6. We conjecture that the other ladder representations can also be realized this way.

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